

A Circle Viewed by Scale Depth

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At first sight, π and e seem to belong to different parts of mathematics. The constant π is geometric: it measures the shape of a circle. The constant e is dynamic: it appears when change is proportional to what is already present. Yet both arise naturally from the elementary area law

$$A(r) = \pi r^2.$$

The connection becomes visible when radius is viewed not as ordinary distance, but as multiplicative scale. In that coordinate, circle area becomes exponential. The same viewpoint also gives a simple continuous law for how measurement error propagates into an estimate of π .

There is an old story behind these two constants. In *Measurement of a Circle*, Archimedes gave rigorous upper and lower bounds for the circular constant,

$$\frac{223}{71} < \pi < \frac{22}{7},$$

using inscribed and circumscribed polygons rather than decimal measurement [1, 2]. Nearly eighteen centuries later, Napier introduced logarithms to turn multiplicative calculation into additive calculation; his original construction was itself described through moving points, one with uniform motion and one whose speed depended on the distance remaining [3, 4]. Euler then placed exponential and logarithmic functions at the center of analysis in the eighteenth century [5, 6]. The small observation below sits naturally along that historical line: geometry, multiplicative scale, and continuous change can all be read in one elementary formula.

Differentiating the area law with respect to r gives

$$\frac{dA}{dr} = 2\pi r.$$

Because $\pi = A/r^2$, this may be written entirely in terms of A and r :

$$\frac{dA}{dr} = \frac{2}{r} A.$$

The derivative is already proportional to the area itself. The only difference from the usual exponential-growth equation is the factor $1/r$.

Now introduce the logarithmic radius

$$t = \ln r.$$

Historically, this is exactly the kind of operation logarithms were built to perform: multiplication in the original variable becomes addition in the logarithmic one. Napier's purpose was computational, but the same structural idea appears here geometrically. Equal multiplicative changes of radius become equal additive steps in t . Then $r = e^t$ and $dr/dt = r$. The chain rule removes the troublesome factor:

$$\frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt} = \frac{2A}{r} r = 2A.$$

Thus the familiar circle law becomes

$$\frac{dA}{dt} = 2A.$$

Its solution is

$$A(t) = Ce^{2t}.$$

By Euler's time, the reciprocal roles of the natural exponential and logarithm had become part of the emerging language of analysis [5, 6]. Here that language does not replace the geometry; it simply gives the geometry a new coordinate. Since $t = \ln r$,

$$A(r) = Ce^{2\ln r} = Cr^2.$$

Comparison with $A(r) = \pi r^2$ gives $C = \pi$, so

$$A(t) = \pi e^{2t}.$$

Nothing about the circle has changed. Only the coordinate has changed. In ordinary radius, area follows the power law $A \propto r^2$; in logarithmic radius, the same law is exponential, $A \propto e^{2t}$. The constant π is the geometric normalization—the area at $t = 0$, where $r = 1$ —while e appears because equal increments of t correspond to equal multiplicative changes of r .

Scale depth

The variable $t = \ln r$ has a useful interpretation. If t increases by one, then

$$t \mapsto t + 1 \quad \implies \quad r \mapsto er.$$

Thus t records multiplicative scale in additive units. It can be read as a continuous *scale depth*: moving an equal distance in t means multiplying the radius by the same factor.

This is the same structural reason logarithms appear in tree depth. If the population of a balanced binary tree grows like 2^d , then its level is $d = \log_2 n$. There is no literal tree hidden in a circle. The common feature is repeated multiplication. A logarithm turns multiplicative growth into additive depth. What Napier introduced as a way to simplify products can therefore be read more broadly as a coordinate for scale: the logarithm records *how many multiplicative levels* separate one magnitude from another.

The circle law itself makes this explicit. Taking logarithms gives

$$\ln\left(\frac{A}{\pi}\right) = 2 \ln r = 2t.$$

A multiplication of radius by k multiplies area by k^2 ; in the logarithmic coordinate, that same change is merely an additive displacement. The nonlinear scaling law has been straightened into a linear one.

Scale depth and precision

The same coordinate gives a simple way to discuss estimation. From

$$A(t) = \pi e^{2t}$$

we may infer π from an area measurement by writing

$$\widehat{\pi}(t) = \widehat{A}(t)e^{-2t}.$$

Suppose first that the area is measured with a fixed absolute error,

$$|\widehat{A}(t) - A(t)| \leq \varepsilon.$$

Then

$$|\widehat{\pi} - \pi| = e^{-2t}|\widehat{A} - A| \leq \varepsilon e^{-2t}.$$

Hence the uncertainty in the inferred value of π obeys the continuous bound

$$|\widehat{\pi} - \pi| \leq P_\pi(t), \quad P_\pi(t) = \varepsilon e^{-2t}.$$

The precision envelope itself satisfies

$$\frac{dP_\pi}{dt} = -2P_\pi.$$

This is the mirror image of the area equation $dA/dt = 2A$: the geometric signal grows like e^{2t} , while a fixed absolute measurement error, after normalization by that signal, contracts like e^{-2t} .

Since $e^{-2t} = 1/r^2$, the same result in ordinary radius is simply

$$|\widehat{\pi} - \pi| \leq \frac{\varepsilon}{r^2}.$$

A larger circle therefore gives a tighter estimate of π *if the absolute uncertainty in measured area remains fixed*. The qualification matters. If instead the measurement has fixed relative error,

$$|\widehat{A} - A| \leq \rho A,$$

then

$$|\widehat{\pi} - \pi| \leq \rho\pi,$$

and increasing the scale gives no improvement. Scale does not create information; it helps only when the uncertainty grows more slowly than the signal.

There is also a clean local relation when both area and scale depth are uncertain. Since

$$\pi = Ae^{-2t},$$

taking logarithms and differentiating gives

$$\frac{d\pi}{\pi} = \frac{dA}{A} - 2dt.$$

Thus, for small errors,

$$\frac{|\Delta\pi|}{\pi} \lesssim \frac{|\Delta A|}{A} + 2|\Delta t|.$$

Because $dt = dr/r$, this is equivalent to

$$\frac{|\Delta\pi|}{\pi} \lesssim \frac{|\Delta A|}{A} + 2\frac{|\Delta r|}{r}.$$

The logarithmic coordinate therefore does more than expose e . It puts geometric growth and uncertainty propagation into the same language.

For finite, rather than infinitesimal, uncertainty one can keep the statement exact. If an observation gives $\hat{A}$ and $\hat{t}$ with

$$|A - \hat{A}| \leq \varepsilon, \quad |t - \hat{t}| \leq \delta,$$

and $\hat{A} > \varepsilon$, then $\pi = Ae^{-2t}$ lies in the interval

$$(\hat{A} - \varepsilon)e^{-2(\hat{t}+\delta)} \leq \pi \leq (\hat{A} + \varepsilon)e^{-2(\hat{t}-\delta)}.$$

This is a continuous precision bound indexed directly by scale depth.

The picture is therefore compact. Ordinary geometry begins with $A = \pi r^2$. Passing to $t = \ln r$ turns multiplicative scale into additive depth, the power law into exponential growth, and fixed absolute measurement uncertainty into exponential error contraction:

$$\frac{dA}{dt} = 2A, \quad \frac{dP_\pi}{dt} = -2P_\pi.$$

The geometry supplies π , the scale coordinate reveals e , and the logarithm makes both growth and precision linear in depth. Archimedes' circle, Napier's logarithm, and Euler's exponential enter for different historical reasons; in this coordinate they meet in the same short calculation.

References

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