A Spark for a Yang–Mills Field

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Start with the field inside a capacitor, let the field interact with itself, and ask what would turn that extra interaction into something we could charge, control, and discharge.

1. The familiar starting point: Maxwell becomes a circuit

A capacitor is already field engineering: separate some charge, and energy sits between the plates. To see what Yang–Mills might add, first recover this familiar device from field theory.

Electromagnetism is the simplest gauge theory. Its potential A_μ describes the field, with $\mu, \nu = 0, 1, 2, 3$ labeling spacetime coordinates and ∂_μ a coordinate derivative. In relativistic units, the field strength and Lagrangian density are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mathcal{L}_{\rm EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu. \tag{1}$$

Here J^μ describes charges and currents. Repeated indices are summed, with Minkowski signature $(+,-,-,-)$. Requiring the spacetime integral of the Lagrangian to be stationary under variations of A_μ gives $\partial_\mu F^{\mu\nu} = J^\nu$. The definition of F supplies the other Maxwell equations.

The potential contains some redundancy. For any scalar function α, replacing A_μ by $A_\mu + \partial_\mu \alpha$ leaves F unchanged. This is a gauge change: two representations of the same physical field. A gauge-invariant quantity has the same value in either representation.

Now return temporarily to SI units and ask how much energy the field stores. For electric and magnetic fields $\mathbf{E}, \mathbf{B}$,

$$H_{\rm EM} = \frac{1}{2} \int_\Omega \left(\epsilon_0 |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu_0} \right) d^3x, \tag{2}$$

where Ω is the field region and ϵ_0, μ_0 are the vacuum permittivity and permeability. Put ideal parallel plates of area S a distance d apart. A voltage U gives $|\mathbf{E}| = U/d$, so, neglecting edge effects,

$$C = \frac{\epsilon_0 S}{d}, \qquad H_C = \tfrac{1}{2} \epsilon_0 (U/d)^2 S d = \tfrac{1}{2} C U^2 = \frac{Q_c^2}{2C}, \tag{3}$$

Here C is capacitance and $Q_c = CU$ is plate charge. The capacitor formula has emerged by adding up field energy. Close a switch through a resistor and it becomes a heat pulse. Charging paid for the energy; the switch chooses when it leaves.

A coil stores magnetic energy $H_L = LI^2/2$, defining inductance L for current I. Connect the two, and energy can move between the capacitor and the coil. When the circuit is small compared with the relevant wavelength, we can summarize its field dynamics by one variable, Q_c, with $I = \dot{Q}_c$:

$$\mathcal{L}_{LC} = \tfrac{1}{2} L \dot{Q}_c^2 - \frac{Q_c^2}{2C}, \qquad L\ddot{Q}_c + \frac{Q_c}{C} = 0, \qquad \omega_0 = \frac{1}{\sqrt{LC}}. \tag{4}$$

Dots denote time derivatives and ω_0 is the angular oscillation frequency. This is a reduction from a field to a circuit: a few variables retain the behavior we want to control [1]. Conducting matter supplies the geometry and switch. Classical vacuum Maxwell theory has no direct photon self-coupling, although matter and quantum corrections can introduce nonlinear effects.

2. The extra ingredient: the force field couples to itself

Yang–Mills theory adds internal field components that can interact with one another. The potentials become matrices: $A_\mu = A_\mu^a T^a$, with Hermitian basis matrices T^a called generators and internal index a. Their commutator, $[T^a, T^b] = T^a T^b - T^b T^a = i f^{abc} T^c$, measures their failure to commute and defines the constants f^{abc}. With coupling strength g and units $\hbar = c = 1$,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \qquad \mathcal{L}_{\rm YM} = -\tfrac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}. \tag{5}$$

The new ingredient is gAA: the field itself contributes to its field strength. Squaring F produces cubic and quartic terms in A, giving interactions involving three and four force particles in the quantum description [2].

Restrict all potentials to one fixed generator, so their matrices commute: the extra term vanishes and the source-free equations become Maxwell's equations. This commuting case is Abelian; the generalization is non-Abelian. Electromagnetic devices also require sources.

3. An exact classical reduction that needs self-interaction

Can this extra term produce a simple energy-storing system? Take the simplest non-Abelian group, $SU(2)$, in a periodic box of volume V. Choose temporal gauge, $A_0^a = 0$, and let the three spatial potentials point along three different internal generators, all with amplitude $\phi(t)$. The field is uniform in space:

$$A_i^a(t) = \phi(t)\delta_i^a, \qquad E_i^a := F_{0i}^a = \dot{\phi}\delta_i^a, \qquad B_i^a := \tfrac{1}{2}\varepsilon_{ijk} F_{jk}^a = g\phi^2 \delta_i^a. \tag{6}$$

Here $i, j, k, a = 1, 2, 3$, δ_i^a is the identity array, ε_{ijk} is the antisymmetric symbol, and $f^{abc} = \varepsilon^{abc}$. Notice the surprise: spatial derivatives vanish, yet a magnetic field remains, generated entirely by the noncommuting potentials. The source-free Gauss constraint is also satisfied: its remaining term is $g\varepsilon^{abc} \sum_i A_i^b E_i^c = 0$.

Adding up the electric and magnetic energies, and reducing the action, gives

$$H_{\rm red} = \frac{3V}{2}\left(\dot{\phi}^2 + g^2\phi^4\right), \qquad \mathcal{L}_{\rm red} = \frac{3V}{2}\left(\dot{\phi}^2 - g^2\phi^4\right), \qquad \boxed{\ddot{\phi} + 2g^2\phi^3 = 0.} \tag{7}$$

The full field equations give this same oscillator equation: solutions starting in this restricted family stay in it. Such consistent reductions and their coupling to waves are established subjects [3]. At a turning point, $\phi = \Phi$ and $\dot{\phi} = 0$, all the energy is magnetic:

$$H_{\rm int} = \tfrac{3}{2} V g^2 \Phi^4. \tag{8}$$

Now remove the commutator. The same uniform, initially stationary potentials have $F = 0$: this magnetic energy and its restoring force disappear. Maxwell fields can of course store energy in other configurations. The comparison isolates what self-interaction contributes here. Although A_i^a and ϕ depend on gauge, the sums $\sum_{i,a}(B_i^a)^2$ and $\sum_{i,a}(E_i^a)^2$, and hence the energy, do not.

We have reached a field analogue of the resonator, with a different rule. The ideal LC circuit has a linear restoring force and fixed frequency. Equation (7) has a cubic restoring force and frequency proportional to $g|\Phi|$: its rhythm changes with its amplitude. Magnetic energy becomes electric energy and back through the field's own interactions. This suggests amplitude-tuned resonators and transfer between field modes as engineering questions. It is an established classical example; its stability against spatially varying disturbances and its quantum regime still need analysis.

An oscillator becomes useful when something can receive its energy. Suppose a collector acts like a damping term with coefficient $\nu(t) \geq 0$. An illustrative model is

$$\ddot{\phi} + \nu(t)\dot{\phi} + 2g^2\phi^3 = 0, \qquad -\dot{H}_{\rm red} = 3V\nu(t)\dot{\phi}^2 =: P_{\rm env}. \tag{9}$$

Here $P_{\rm env}$ is power transferred to the environment; only a useful fraction reaches a load. One possible gauge-invariant coupling is $H_{\rm couple} = \lambda(t)X_{\rm col}\int_\Omega \sum_{i,a}(B_i^a)^2 d^3x$, where $X_{\rm col}$ is a gauge-neutral collector observable and λ carries the required units. Deriving damping, building the collector, and paying for switching remain separate tasks. QCD would also require handling confinement and coupling to ordinary matter.

4. From an oscillator to a field that waits for a spark

The oscillator stores energy, but it does not sit quietly awaiting a spark: the quartic potential has only one minimum, at $\phi = 0$. For fuel-like behavior, think instead of a click-activated hand warmer. Its contents wait in a metastable state—a local energy minimum—until a small disturbance starts a larger rearrangement.

Assume a four-dimensional quantum theory exists with energy operator, or Hamiltonian, H. Density operators $\rho_{\rm fuel}$ and $\rho_{\rm spent}$ describe the initial and final physical states. The trace averages energy over each state, so the energy released is

$$Q = {\rm Tr}(H\rho_{\rm fuel}) - {\rm Tr}(H\rho_{\rm spent}) > 0. \tag{10}$$

An existence theorem establishes a mathematical setting for this question [4]. Finding these states, preparing them, and making one survive until triggered requires additional physics.

To describe what we would need, let $q(x)$ summarize the local field state at position x. It must be dimensionless and gauge invariant; a normalized, spatially averaged magnetic energy density is one possible starting point. Assign the spent and fuel branches $q = 0$ and $q = 1$. This is a physical collective variable, distinct from the gauge-dependent amplitude ϕ. Suppose averaging over the other variables gives

$$E[q] = \int_\Omega \left[\frac{\kappa}{2}|\nabla q|^2 + u(q)\right] d^3x, \qquad u(q) = aq^2(1-q)^2 + \epsilon(3q^2 - 2q^3), \quad a > 3\epsilon > 0. \tag{11}$$

The function u is an energy landscape with two valleys. The coefficient $\kappa > 0$ makes neighboring regions prefer similar values of q; a, ϵ are energy densities. Both uniform states sit at local minima, since $u''(0) = 2(a + 3\epsilon) > 0$ and $u''(1) = 2(a - 3\epsilon) > 0$. The fuel valley is higher by ϵ, giving $Q = \epsilon V$. The hill between them peaks at $q = (a + 3\epsilon)/(2a)$.

This landscape is the missing bridge: Eq. (7) does not derive Eq. (11). We would have to obtain it from a specified gauge theory and environment. To attribute the mechanism to force-particle interactions, we should also repeat the calculation with the noncommuting sector removed, keeping the physical control constraints fixed. Its proposed storage channel or ignition mechanism should then disappear. The oscillator passes this *Abelian comparison* for its magnetic channel; the two-valley fuel model has yet to earn that connection.

If those valleys exist, a local spark could create a small bubble of the spent state inside the fuel. Its boundary costs energy, while its converted interior releases energy. For radius R and interface tension σ (energy per area), the balance is

$$E_{\rm seed}(R) \simeq 4\pi\sigma R^2 - \frac{4\pi}{3}\epsilon R^3, \qquad R_* = \frac{2\sigma}{\epsilon}, \qquad E_* = \frac{16\pi\sigma^3}{3\epsilon^2}. \tag{12}$$

This approximation requires a wall thin compared with R. Small bubbles shrink; beyond the critical radius R_*, further growth lowers the static energy. The barrier height is E_*. When the valleys differ only slightly in depth, $\epsilon/a \ll 1$, their interface tension is approximately

$$\sigma \simeq \int_0^1 \sqrt{2\kappa a q^2 (1-q)^2}\, dq = \frac{\sqrt{2\kappa a}}{6}. \tag{13}$$

These formulas locate a static barrier. Quantum tunneling rates [5] and the work needed to create a seed require further calculations. A local control could add $-\int_\Omega j(t,x)q(x)d^3x$ to the Hamiltonian: j is a control energy density, and a negative pulse favors $q = 0$. Its physical implementation remains to be found. If seeds form at approximately constant rates $\Gamma_{\rm off}$ without control and $\Gamma_{\rm on}$ with it, we need

$$\Gamma_{\rm off} t_{\rm store} \ll 1, \qquad \Gamma_{\rm on} t_{\rm trigger} \gg 1. \tag{14}$$

In words: almost no spontaneous seeds during storage time $t_{\rm store}$, but very likely a seed during trigger time $t_{\rm trigger}$. After ignition, the conversion front still needs time to cross the sample.

5. The same energy ledger, with a new possible mechanism

A spark releases stored energy, as a switch discharges a capacitor. Count what reaches the load, $W_{\rm load}$; net actuator work into the field and collector, $W_{\rm drive}$; and uncollected energy, $E_{\rm loss} \geq 0$. Let $C_{\rm acq}$ be acquisition and handling costs in energy units, outside that subsystem. With control off at both endpoints,

$$W_{\rm load} + E_{\rm loss} = Q + W_{\rm drive}, \qquad W_{\rm net} := W_{\rm load} - W_{\rm drive} - C_{\rm acq} = Q - E_{\rm loss} - C_{\rm acq}. \tag{15}$$

A larger batch holds more energy, but offers more places for premature ignition. Write $E_{\rm loss} = (1-\eta)Q + L_{\rm act}$, with collection efficiency $0 < \eta \leq 1$ and fixed actuation loss $L_{\rm act} \geq 0$. For spontaneous seeds forming as a Poisson process with rate γ per unit volume, survival probability is $e^{-\gamma V t_{\rm store}}$. Positive output and survival probability at least $p \in (0,1)$ require

$$\boxed{\frac{C_{\rm acq} + L_{\rm act}}{\eta\epsilon} < V \leq \frac{-\log p}{\gamma t_{\rm store}}.} \tag{16}$$

The batch must repay its costs (left bound) and survive storage (right bound). We assume fixed $C_{\rm acq}, L_{\rm act}, \eta, \epsilon, \gamma$ over this range; the seed must also fit inside. Resetting using work alone costs at least Q, preventing positive net energy from a complete closed cycle.

The engineering pattern is familiar: charge, control, collect. Yang–Mills adds a concrete self-interaction and a different oscillator. Making that field wait for a spark remains the intriguing step: derive the two valleys, find the trigger, and follow the released energy.

References

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