The Precision Staircase Hidden in Euler's Number

A short note on inverse-factorial convergence and digit gain

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Main framing and innovation

The elementary series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

is usually introduced as a convenient way to compute Euler's number. This note reframes the same series as a *precision-generating mechanism*. The main object is a digit-level invariant

$$\mathcal{I}(\alpha) = \log_{10}\left(\frac{(\alpha+1)^3}{\alpha(\alpha+2)}\right),$$

which measures the latent decimal precision gained by adding the α-th term. The novelty here is not the classical expansion itself, but the compact precision geometry it exposes: the approximation to e does not merely converge; it climbs an accelerating staircase of guaranteed decimal information.

1 Euler's number as a self-replicating function

A clean modern entry point to Euler's number is differential rather than numerical. Define e^x as the unique function satisfying

$$\frac{d}{dx}e^x = e^x, \qquad e^0 = 1.$$

Euler's number is then the value of this function at one:

$$e = e^1.$$

The Maclaurin expansion of the exponential function gives

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This representation is standard, going back to Euler's systematic treatment of the exponential and logarithmic functions and appearing in modern references such as the DLMF [1, 2]. The point of this note is to look at that familiar expansion from a slightly different angle. Instead of asking only how close the partial sums are, we ask how many *decimal facts* about e each new term certifies.

2 The precision staircase

Let

$$S_\alpha = \sum_{n=0}^{\alpha} \frac{1}{n!}, \qquad \alpha \in \mathbb{N} \cup \{0\},$$

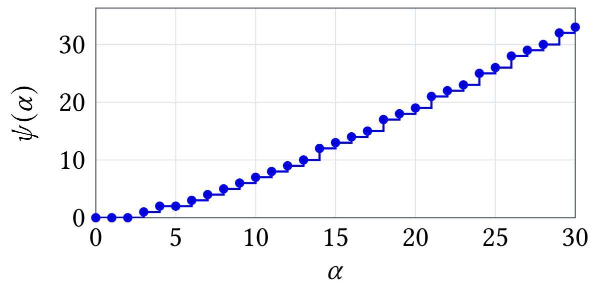

Figure 1: The guaranteed decimal precision $\psi(\alpha)$ is integer-valued, so it forms a staircase.

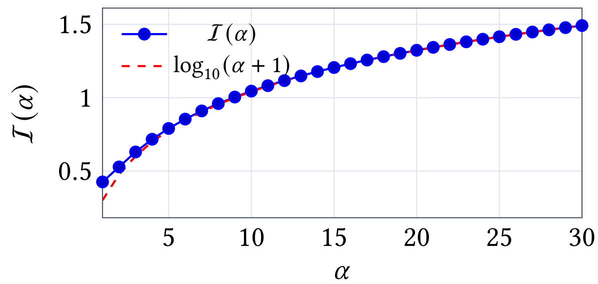

Figure 2: The latent gain is smooth and asymptotic to $\log_{10}(\alpha + 1)$.

be the truncated inverse-factorial approximation to e. Its remainder is

$$R_\alpha = \mathrm{e} - S_\alpha = \sum_{n=\alpha+1}^{\infty} \frac{1}{n!}.$$

Pulling out the first omitted term gives

$$R_\alpha = \frac{1}{(\alpha+1)!}\left(1 + \frac{1}{\alpha+2} + \frac{1}{(\alpha+2)(\alpha+3)} + \cdots\right).$$

The bracketed expression is bounded above by a geometric series, because every new denominator factor is at least $\alpha + 2$. Hence

$$R_\alpha < \frac{1}{(\alpha+1)!}\left(1 + \frac{1}{\alpha+2} + \frac{1}{(\alpha+2)^2} + \cdots\right) = \frac{\alpha+2}{(\alpha+1)(\alpha+1)!}.$$

Define

$$B_\alpha = \frac{\alpha+2}{(\alpha+1)(\alpha+1)!}.$$

The bound $R_\alpha < B_\alpha$ turns the convergence of the series into a decimal scale. If $B_\alpha \leq 10^{-\phi}$, then S_α certifies ϕ decimal places in absolute error. This motivates the following function.

Definition 1: Guaranteed decimal precision

For $\alpha \geq 0$, define

$$\boxed{\psi(\alpha) = \max\left\{0, \left\lfloor \log_{10}\left(\frac{(\alpha+1)(\alpha+1)!}{\alpha+2}\right)\right\rfloor\right\}.}$$

Equivalently, $\psi(\alpha) = \max\{0, \lfloor -\log_{10} B_\alpha \rfloor\}$.

The floor function is doing something conceptually important. The error bound itself decays smoothly, but decimal knowledge is discrete. A digit appears only when the tail drops below the next power of ten. This creates the *precision staircase*: flat steps when the hidden gain has not crossed a decimal threshold, and jumps when it has.

For instance, $\psi(10) = 7$, while $\psi(20) = 19$. Thus the twentieth truncation is not just twice as good as the tenth in any linear sense; the factorial tail has collapsed the uncertainty by many additional orders of magnitude. This kind of reasoning is part of the broader culture of truncation error and numerical reliability in scientific computation [3, 4].

3 The digit-gain invariant

The staircase is useful, but it hides the underlying smooth quantity. Define the *latent precision*

$$P(\alpha) = \log_{10}\left(\frac{(\alpha+1)(\alpha+1)!}{\alpha+2}\right),$$

so that $\psi(\alpha) = \max\{0, \lfloor P(\alpha) \rfloor\}$. Then the natural unit of analysis is the precision gained by moving from $S_{\alpha-1}$ to S_α, i.e. by adding the α-th term.

Definition 2: Precision-gain invariant

For $\alpha \geq 1$, define

$$\boxed{\mathcal{I}(\alpha) = P(\alpha) - P(\alpha-1).}$$

Proposition

For $\alpha \geq 1$, the precision-gain invariant has the closed form

$$\boxed{\mathcal{I}(\alpha) = \log_{10}\left(\frac{(\alpha+1)^3}{\alpha(\alpha+2)}\right).}$$

Moreover,

$$\mathcal{I}(\alpha) = \log_{10}(\alpha+1) - \log_{10}\left(1 - \frac{1}{(\alpha+1)^2}\right),$$

so

$$\mathcal{I}(\alpha) \sim \log_{10}(\alpha+1) \qquad (\alpha \to \infty).$$

Proof. By definition,

$$P(\alpha) = \log_{10}\left(\frac{(\alpha+1)(\alpha+1)!}{\alpha+2}\right),$$

and

$$P(\alpha-1) = \log_{10}\left(\frac{\alpha\alpha!}{\alpha+1}\right).$$

Therefore

$$\mathcal{I}(\alpha) = \log_{10}\left(\frac{\frac{(\alpha+1)(\alpha+1)!}{\alpha+2}}{\frac{\alpha\alpha!}{\alpha+1}}\right).$$

Using $(\alpha+1)! = (\alpha+1)\alpha!$, this becomes

$$\mathcal{I}(\alpha) = \log_{10}\left(\frac{(\alpha+1)^3}{\alpha(\alpha+2)}\right).$$

Finally, since $\alpha(\alpha+2) = (\alpha+1)^2 - 1$,

$$\frac{(\alpha+1)^3}{\alpha(\alpha+2)} = \frac{\alpha+1}{1-(\alpha+1)^{-2}},$$

which gives the asymptotic relation. □

This invariant explains the bumpiness of ψ. The realized integer gain is

$$\Delta\psi(\alpha) = \psi(\alpha) - \psi(\alpha - 1),$$

while the latent gain is $\mathcal{I}(\alpha)$. More explicitly,

$$\Delta\psi(\alpha) = \lfloor P(\alpha - 1) + \mathcal{I}(\alpha) \rfloor - \lfloor P(\alpha - 1) \rfloor .$$

Thus $\mathcal{I}(\alpha)$ is the continuous pressure pushing the staircase upward, and the floor function is the gate that turns that pressure into visible digit jumps.

4 Interpretation: a hidden elegance in e

Euler's number is often praised for its appearances: compound interest, natural growth, differential equations, complex rotation, and the identity $\mathrm{e}^{i\pi} + 1 = 0$. Here we see another elegance, quieter but very precise. The inverse-factorial series for e is not merely a sequence of approximations. It is a machine for converting terms into decimal certainty.

The important conceptual point is that the convergence is not steady. Each new term arrives after a factorial denominator has grown, and that factorial growth makes the remaining uncertainty collapse at an accelerating rate. In ordinary language, later terms do not just refine the approximation; they purchase increasingly many digits at once.

The invariant makes this vivid:

$$\mathcal{I}(\alpha) \sim \log_{10}(\alpha + 1).$$

At term α, the number of latent decimal digits gained is roughly the number of digits in $\alpha + 1$ itself. That is the hidden rhythm: as the index grows by ordinary counting, the certified information grows by logarithmic digit gain. The series is therefore simultaneously simple and deeply efficient.

What has been learned

The same number defined by the self-replication law $(\mathrm{e}^x)' = \mathrm{e}^x$ also carries a self-tightening computational structure. Its Taylor series does not simply approach e; it organizes the approach into an accelerating hierarchy of decimal precision. In that sense, Euler's number contains a small precision geometry: a staircase of visible digits driven by a smooth invariant underneath.

Acknowledgment of scope

The estimates used above are elementary consequences of the classical exponential series. The contribution of this note is the framing: naming the latent termwise digit gain, isolating its closed form, and using it to explain the precision staircase produced by truncating the series for e.

References

[1] L. Euler, *Introductio in analysin infinitorum*, Marc-Michel Bousquet, Lausanne, 1748.

[2] NIST Digital Library of Mathematical Functions, "Chapter 4: Elementary Functions," especially §4.6, Power Series. Release 1.2.7, 2026. DLMF §4.6.

[3] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002.

[4] D. E. Knuth, *The Art of Computer Programming, Volume 2: Seminumerical Algorithms*, 3rd ed., Addison-Wesley, 1998.

[5] E. Maor, *e: The Story of a Number*, Princeton University Press, 1994.

[6] J. J. O'Connor and E. F. Robertson, "The number e," MacTutor History of Mathematics Archive, 2001. MacTutor.