# Gaussian Halfspace Testing, Distributional Stability, and the Exact Lognormal Pullback

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**Abstract**

Recent work of Chen, De, Huang, Nadimpalli, Servedio, and Yang shows that halfspaces admit sublinear-query relative-error testing under the standard Gaussian law, in sharp contrast with the harder Boolean-cube setting. This paper isolates a simple transport viewpoint for such results. First, if a tester depends on the sampling distribution only through $q$ independent draws, then changing the ambient law by total variation distance $\Delta$ changes its acceptance probability by at most $q\Delta$, so any completeness–soundness gap degrades by at most $2q\Delta$. Second, if a bimeasurable change of variables pushes an input law $D$ exactly to $N(0, I_n)$, then the Gaussian testing problem transfers unchanged to the corresponding pullback class under $D$. For multivariate lognormal data, taking coordinatewise logarithms and whitening yields exactly this situation, so Gaussian halfspace testing becomes exact testing of log-halfspaces. The point is not to construct a new tester, but to make clear how Gaussian testing results behave under near-exact and exact changes of coordinates.

*Keywords:* halfspace testing, relative-error testing, Gaussian space, distribution shift, lognormal distribution

## 1 Introduction

Let

$$\mathcal{H} := \left\{ h_{w,\theta}(x) = \mathbf{1}[w^\top x \geq \theta] : w \in \mathbb{R}^n,\ \theta \in \mathbb{R} \right\}$$

be the class of halfspaces on $\mathbb{R}^n$. In the relative-error model, the distance from a Boolean function $f \colon \mathbb{R}^n \to \{0, 1\}$ to a comparator $h$ under a law $\mu$ is

$$d_\mu^{\mathrm{rel}}(f, h) := \frac{\mu(f \triangle h)}{\mu(f = 1)},$$

where $f \triangle h$ denotes the disagreement set. This normalization makes sparse positive regions expensive to ignore, and for that reason it behaves differently from the standard additive-error model.

Figure 1 makes the geometry of this normalization explicit. The numerator is the $\mu$-mass of the disagreement region, while the denominator is the $\mu$-mass of the positive set itself. When the positive region is small, even visually modest disagreement can carry a large relative penalty after this renormalization.

A recent preprint of Chen et al. (2026a) shows that halfspaces are relative-error testable under the standard Gaussian law $G := N(0, I_n)$ with query complexity sublinear in the ambient dimension. That result sits in an interesting landscape. In the classical property-testing model, halfspaces are testable with query complexity independent of $n$ (Matulef et al., 2010). In the relative-error model over the Boolean cube, by contrast, halfspaces already require $\widetilde{\Omega}(\log n)$

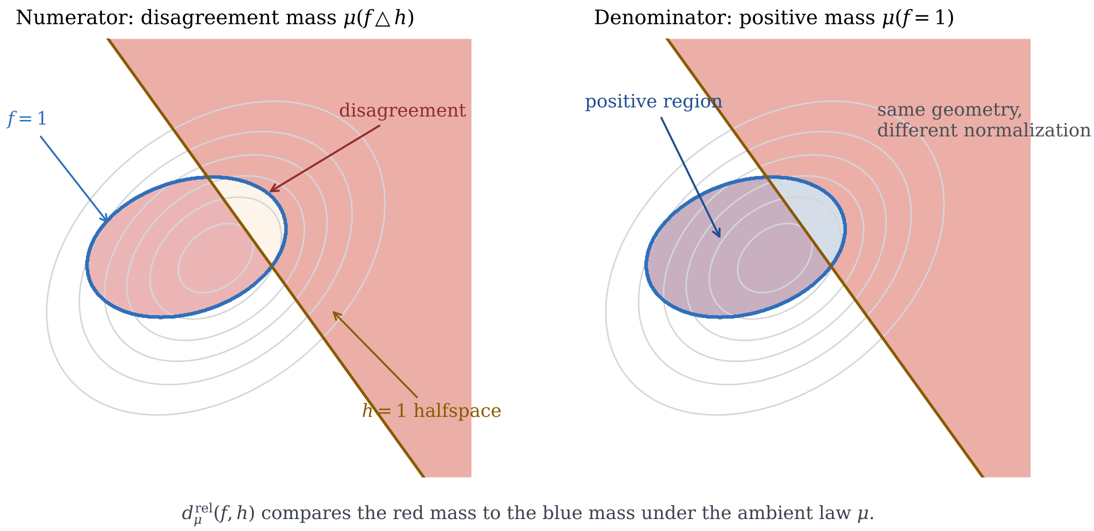


Figure 1: A geometric view of relative error under an ambient law $\mu$. The same classifier geometry is shown twice: the red region carries the disagreement mass $\mu(f \triangle h)$, while the blue region carries the positive mass $\mu(f=1)$. Relative error compares these two weighted quantities rather than measuring raw disagreement alone.

queries (Chen et al., 2026b); see also Chen et al. (2025) for one of the early positive results in the same model.

The question here is slightly different. Suppose the ambient distribution is not Gaussian in its native coordinates, but becomes Gaussian after a natural change of variables. How much of the Gaussian theorem survives? The answer comes in two layers.

First, there is a simple algorithmic stability lemma: if a tester interacts with the sampling law only through $q$ independent draws, then changing the law by total variation $\Delta$ perturbs acceptance probabilities by at most $q\Delta$. This is the elementary robustness statement.

Second, in the exact case, one gets more than robustness. If a bimeasurable map $T$ pushes $D$ exactly to $G$, then the testing problem under $D$ is literally the Gaussian problem written in different coordinates. The lognormal family is a clean example: after taking logs and whitening, one lands exactly in Gaussian space. The resulting pullback class is the family of log-halfspaces.

## 2 A $q\Delta$ transfer lemma

For this section it is enough to work with a generic sample-based tester. Fix a probability measure $\mu$ on a measurable space $\mathcal{X}$, and let $f\colon \mathcal{X} \to \{0,1\}$ be the target function. A $q$-sample tester is a randomized measurable rule that receives $q$ i.i.d. labeled samples

$$(X_1, f(X_1)), \dots, (X_q, f(X_q)), \qquad X_i \sim \mu,$$

and then outputs ACCEPT or REJECT. All internal computation is allowed; the only point is that the ambient law enters through at most $q$ independent draws.

Write $\Pr_\mu[\mathcal{A}^f = 1]$ for the acceptance probability of such a tester under the sampling law $\mu$.

**Theorem 1** (Instancewise stability under a change of measure)**.** *Let $\mathcal{A}$ be any $q$-sample tester, and let $\mu$ and $\nu$ be probability measures on $\mathcal{X}$. Then for every Boolean target function $f$,*

$$\left| \Pr_\mu[\mathcal{A}^f = 1] - \Pr_\nu[\mathcal{A}^f = 1] \right| \le q\, \|\mu - \nu\|_{\mathrm{TV}}.$$

*Proof.* Set $\Delta := \|\mu - \nu\|_{\mathrm{TV}}$. By the coupling characterization of total variation distance, there exists a coupling $(X, Y)$ with marginals $\mu$ and $\nu$ such that $\Pr[X \neq Y] \leq \Delta$. Take $q$ independent copies $(X_i, Y_i)$ of this coupling and run the tester with the same internal randomness on the two labeled sample sequences

$$(X_1, f(X_1)), \ldots, (X_q, f(X_q)) \quad \text{and} \quad (Y_1, f(Y_1)), \ldots, (Y_q, f(Y_q)).$$

If $X_i = Y_i$ for every $i$, then the two executions see identical inputs and therefore produce the same output. Hence the acceptance indicators can differ only on the event $\{\exists i : X_i \neq Y_i\}$, whose probability is at most $q\Delta$ by the union bound. Taking expectations gives the claim. □

The theorem immediately converts any completeness–soundness guarantee into a perturbed one for the same promise problem.

**Corollary 2** (Gap transfer for fixed promise sets)**.** *Let $\mathcal{Y}$ and $\mathcal{N}$ be two families of Boolean functions on $\mathcal{X}$. Assume a $q$-sample tester $\mathcal{A}$ satisfies*

$$\Pr_{\mu}[\mathcal{A}^f = 1] \geq c \quad \text{for all } f \in \mathcal{Y}, \qquad \Pr_{\mu}[\mathcal{A}^f = 1] \leq s \quad \text{for all } f \in \mathcal{N},$$

*with gap $\gamma := c - s > 0$. Then under any other sampling law $\nu$ the same tester satisfies*

$$\Pr_{\nu}[\mathcal{A}^f = 1] \geq c - q\|\mu - \nu\|_{\mathrm{TV}} \quad (f \in \mathcal{Y}),$$

$$\Pr_{\nu}[\mathcal{A}^f = 1] \leq s + q\|\mu - \nu\|_{\mathrm{TV}} \quad (f \in \mathcal{N}),$$

*and therefore retains gap at least*

$$\gamma - 2q\|\mu - \nu\|_{\mathrm{TV}}.$$

**Corollary 3** (Constant-overhead amplification)**.** *Assume in Corollary 2 that $c = 2/3$ and $s = 1/3$. If $q\|\mu - \nu\|_{\mathrm{TV}} \leq 1/12$, then one run of the tester under $\nu$ has completeness at least $7/12$ and soundness at most $5/12$. Consequently, a constant number of independent repetitions restores a $2/3$ versus $1/3$ guarantee under $\nu$, so the query complexity remains $O(q)$.*

*Proof.* The one-run bounds are immediate from Corollary 2. Since the two acceptance probabilities are then separated by a positive constant, standard Chernoff amplification yields a fixed repetition count that restores a $2/3$ versus $1/3$ gap. □

*Remark* 4. The point of Theorem 1 is deliberately modest. It controls the *algorithmic* effect of changing the sampling law. If the promise set itself is defined using the ambient measure, as in relative-error testing, then an approximate transfer of a full soundness statement may require a separate comparison between the two induced distance notions. In sparse regimes that extra comparison can be delicate. The exact pullback result below avoids this issue entirely because the underlying testing problem is then identical after conjugation by the transport map.

## 3 Exact pullbacks to Gaussian space

Now let $D$ be a probability measure on a measurable space $\mathcal{X}$, and let

$$T : \mathcal{X} \to \mathbb{R}^n$$

be a bimeasurable bijection. Write $\widetilde{D} := T_{\#}D$ for the pushforward of $D$ by $T$, and for a Boolean function $F : \mathcal{X} \to \{0, 1\}$ define its transported version

$$\widetilde{F}(z) := F(T^{-1}(z)), \qquad z \in \mathbb{R}^n.$$

Also define the pullback halfspace class

$$T^*\mathcal{H} := \{h \circ T : h \in \mathcal{H}\}.$$

When $\widetilde{D} = G$, the testing problem under $D$ is exactly the Gaussian problem in transported coordinates.

**Proposition 5** (Exact pullback principle). *Assume $\widetilde{D} = G = N(0, I_n)$. Then:*

1. *For every $F\colon \mathcal{X} \to \{0,1\}$, one has $F \in T^*\mathcal{H}$ if and only if $\widetilde{F} \in \mathcal{H}$.*
2. *For every $h \in \mathcal{H}$,*

$$D(F \triangle (h \circ T)) = G(\widetilde{F} \triangle h) \quad \textit{and} \quad D(F = 1) = G(\widetilde{F} = 1).$$

*Hence*

$$d_D^{\mathrm{rel}}(F, h \circ T) = d_G^{\mathrm{rel}}(\widetilde{F}, h).$$

3. *Any relative-error tester for $\mathcal{H}$ under $G$ transfers verbatim to a tester for $T^*\mathcal{H}$ under $D$ with the same query complexity, completeness, and soundness.*

*Proof.* The first item is immediate from the definition of $\widetilde{F}$ and the fact that $T$ is invertible.

For the second, let $A \subseteq \mathbb{R}^n$ be measurable. Since $\widetilde{D} = T_\# D = G$, one has

$$D(T^{-1}(A)) = G(A).$$

Apply this identity to the sets

$$A = \{z : \ \widetilde{F}(z) \neq h(z)\} \quad \text{and} \quad A = \{z : \ \widetilde{F}(z) = 1\}.$$

Because $\widetilde{F}(z) = F(T^{-1}(z))$, these are exactly the transported versions of

$$\{x : \ F(x) \neq h(T(x))\} \quad \text{and} \quad \{x : \ F(x) = 1\},$$

which gives the claimed equalities of probabilities and therefore the equality of relative distances.

For the third item, transport every oracle interaction through $T$. A point query to $\widetilde{F}$ at $z$ becomes a query to $F$ at $T^{-1}(z)$. If the tester uses positive examples, then for any measurable $A \subseteq \mathbb{R}^n$,

$$\begin{aligned} T_\#\big(D(\,\cdot\,|F = 1)\big)(A) &= D(T^{-1}(A) \mid F = 1) \\ &= \frac{D(T^{-1}(A) \cap \{F = 1\})}{D(F = 1)} \\ &= \frac{G(A \cap \{\widetilde{F} = 1\})}{G(\widetilde{F} = 1)} \\ &= G(A \mid \widetilde{F} = 1). \end{aligned}$$

So the positive-example oracle also matches exactly after transport. Every transcript under $D$ is therefore identical in law to the corresponding transcript under $G$, which means completeness, soundness, and query complexity are unchanged. □

The proposition is most useful when $T$ has a natural statistical interpretation. The lognormal family is the cleanest such case, and Figure 2 shows the geometry in two dimensions: a curved decision boundary in the original variables becomes an ordinary straight halfspace after logarithmic transport and whitening.

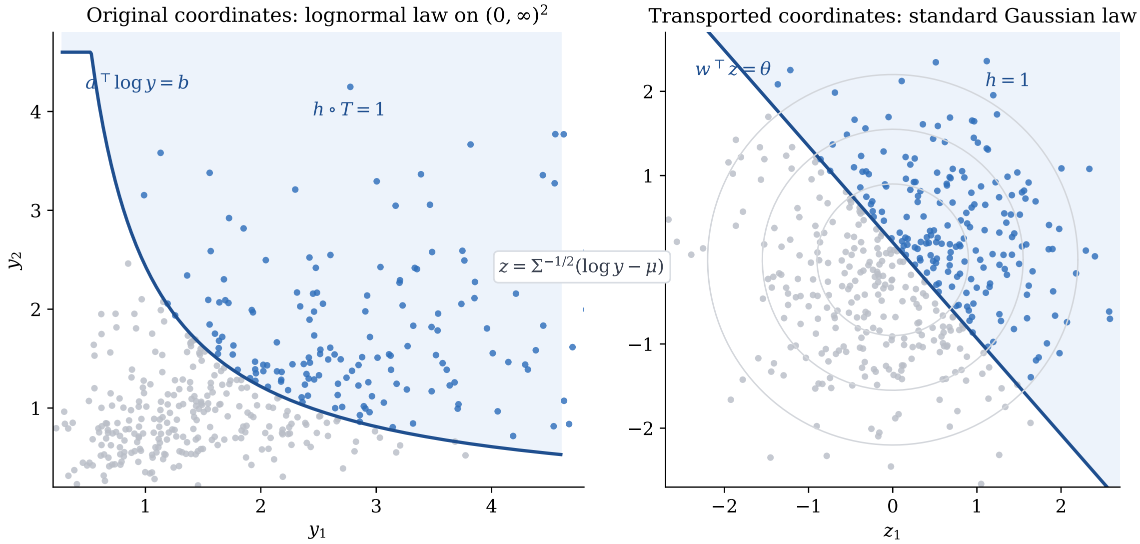


Figure 2: Exact transport from lognormal coordinates to Gaussian coordinates. In the original variables, the pullback of a Gaussian halfspace is a curved log-halfspace boundary. After the map $z = \Sigma^{-1/2}(\log y - \mu)$, the sampling law becomes standard Gaussian and the same decision rule becomes a flat halfspace boundary.

**Corollary 6** (Exact lognormal pullback)**.** *Let* $Y \sim \mathrm{LogNormal}(\mu, \Sigma)$ *on* $(0,\infty)^n$*, where* $\mu \in \mathbb{R}^n$ *and* $\Sigma$ *is positive definite. Define*

$$T(y) := \Sigma^{-1/2}(\log y - \mu),$$

*where the logarithm is taken coordinatewise and* $\Sigma^{-1/2}$ *is the symmetric inverse square root. Then* $T_{\#}D = N(0, I_n)$*, and*

$$T^*\mathcal{H} = \left\{ y \mapsto \mathbf{1}[a^\top \log y \geq b] : a \in \mathbb{R}^n,\ b \in \mathbb{R} \right\}.$$

*Consequently, every Gaussian relative-error tester for halfspaces transfers exactly to a tester for log-halfspaces under the multivariate lognormal law, with identical query complexity and success parameters.*

*Proof.* By definition of the multivariate lognormal law, $\log Y \sim N(\mu, \Sigma)$. Whitening by $\Sigma^{-1/2}$ therefore yields $T(Y) \sim N(0, I_n)$.

For the class identity, note that

$$h_{w,\theta}(T(y)) = \mathbf{1}\left[ w^\top \Sigma^{-1/2}(\log y - \mu) \geq \theta \right] = \mathbf{1}\left[ a^\top \log y \geq b \right],$$

where $a := \Sigma^{-1/2} w$ and $b := \theta + a^\top \mu$. Every log-halfspace arises in this way, so the displayed class is exactly $T^*\mathcal{H}$. The final statement is now Proposition 5. □

# 4 Closing remark

The paper makes three compact points. First, it isolates a general stability mechanism for sample-based testing: when the ambient law is perturbed by total variation $\Delta$, a tester using $q$ independent draws changes its acceptance behavior by at most order $q\Delta$. This is the basic continuity statement behind the discussion, and it explains why Gaussian algorithms should persist under small distributional shifts without needing to be redesigned from scratch.

Second, this work separates the approximate regime from the exact one. A nearby law only gives a controlled loss in gap, but an exact transport to Gaussian space gives something stronger: the testing problem is conjugate to the original Gaussian problem. Membership in the target class, disagreement probabilities, and the relative-error normalization are all preserved under the pullback. That distinction is the real structural message of the paper. Robustness says the theorem survives; exact transport says it has not changed.

Third, the lognormal example shows that this exact viewpoint is not artificial. For multivariate lognormal inputs, logarithmic coordinates followed by whitening place the data exactly in standard Gaussian space, and halfspaces become log-halfspaces in the original variables. So the Gaussian result of Chen et al. (2026a) extends to that setting with no asymptotic penalty and no distortion of the promise problem. In that sense, the lognormal pullback is the natural endpoint of the narrative begun in the introduction: not merely that Gaussian halfspace testing is stable under a useful change of variables, but that in an important class of models the change of variables reveals the same geometric problem underneath.

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