The Formal Boundaries of Solar-System Escape in a Spatial Phase-Torus Model

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1. The Question

The problem is to leave the planetary region as quickly as possible in a spatial, three-dimensional solar-system model. A spacecraft starts near Earth, chooses a launch time, chooses the direction of a fixed-magnitude launch kick, thrusts with bounded acceleration, and may use planetary gravity assists. In this note "exit" means crossing the sphere

$$\|r\| = R_{\text{exit}}, \qquad R_{\text{exit}} > \max_i R_i,$$

while moving outward:

$$r(T) \cdot v(T) > 0.$$

The model is still simplified: planets move on circular inclined orbits, the Sun is the central gravitating body, and flybys are treated by patched-conic jump maps. The central geometric feature is that a spatial flyby is cone-valued: the turn has both a magnitude and an azimuth, so an assist can reshape energy and orbital plane at the same encounter.

2. Planetary Motion on a Spatial Phase Torus

Let planet i have circular orbital radius R_i, gravitational parameter μ_i, initial phase ϕ_i, inclination I_i, and longitude of ascending node Ω_i. Its angular phase is

$$\theta_i(t) = n_i t + \phi_i \pmod{2\pi},$$

with

$$n_i = \sqrt{\frac{\mu_\odot}{R_i^3}},$$

where $\mu_\odot$ is the Sun's gravitational parameter. The joint planetary phase is

$$\Theta(t) = (\theta_1(t), \dots, \theta_m(t)) \in \mathbb{T}^m.$$

Choosing the launch date is therefore choosing

$$\Theta_0 = \Theta(t_0).$$

A single cycle would require a period P such that

$$\Theta(t + P) = \Theta(t) \qquad \text{for all } t,$$

which occurs only when all frequency ratios n_i/n_j are rational. In the generic case, the motion winds on $\mathbb{T}^m$, continually changing the relative alignment of the planets.

Let

$$R_z(\Omega) = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_x(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix}.$$

The embedding of planet i's circular orbit into space is

$$\mathcal{R}_i = R_z(\Omega_i) R_x(I_i).$$

Its position and velocity are

$$r_i(t) = \mathcal{R}_i R_i \begin{pmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \\ 0 \end{pmatrix},$$

and

$$v_i(t) = \mathcal{R}_i R_i n_i \begin{pmatrix} -\sin\theta_i(t) \\ \cos\theta_i(t) \\ 0 \end{pmatrix}.$$

3. Spacecraft Dynamics

The spacecraft state is

$$x(t) = (r(t), v(t)) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

At launch from Earth, with fixed launch excess speed V_0, choose a launch direction $q_0 \in \mathbb{S}^2$. Then

$$r(t_0) = r_\oplus(t_0),$$

$$v(t_0) = v_\oplus(t_0) + V_0 q_0, \qquad \|q_0\| = 1.$$

Between flybys, the dynamics are

$$\dot{r} = v,$$

$$\dot{v} = -\mu_\odot \frac{r}{\|r\|^3} + a_{\max} s(t), \qquad s(t) \in \mathbb{S}^2.$$

Here $s(t)$ is the thrust steering direction. Equivalently, one may write

$$s(t) = \begin{pmatrix} \cos\beta(t)\cos\gamma(t) \\ \sin\beta(t)\cos\gamma(t) \\ \sin\gamma(t) \end{pmatrix},$$

where $\beta(t)$ is an azimuth and $\gamma(t)$ is an elevation. If the engine can coast or throttle, replace the thrust term by

$$\lambda(t) a_{\max} s(t), \qquad 0 \le \lambda(t) \le 1.$$

The thrust acceleration vector is

$$u(t) = a_{\max} s(t), \qquad \|u(t)\| = a_{\max}.$$

4. Spatial Gravity-Assist Map

A flyby of planet p_j at time t_j imposes

$$r(t_j) = r_{p_j}(t_j).$$

Let

$$w^- = v^- - v_{p_j}(t_j)$$

be the incoming planet-relative velocity. A patched-conic flyby preserves relative speed:

$$\|w^+\| = \|w^-\| = v_{\infty,j}.$$

The outgoing relative velocity lies on a cone:

$$\angle(w^+, w^-) = \delta_j, \qquad 0 \le \delta_j \le \delta_{\max,j},$$

with an azimuth ψ_j around the axis w^-.

To parameterize this, define

$$e_1 = \frac{w^-}{\|w^-\|},$$

and choose any orthonormal pair e_2, e_3 spanning the plane perpendicular to e_1. Then

$$w^+ = v_{\infty,j} \left(\cos\delta_j \, e_1 + \sin\delta_j (\cos\psi_j \, e_2 + \sin\psi_j \, e_3)\right).$$

The outgoing heliocentric velocity is

$$v^+ = v_{p_j}(t_j) + w^+.$$

The maximum turn angle is still governed by hyperbolic flyby geometry:

$$\delta_{\max,j} = 2 \arcsin\left(\frac{1}{1 + \frac{r_{p,j}^{\min} v_{\infty,j}^2}{\mu_{p_j}}}\right).$$

Here $r_{p,j}^{\min}$ is the minimum allowed flyby periapsis radius and μ_{p_j} is the gravitational parameter of the flyby planet.

5. The Spatial Minimum-Time Problem

For a flyby route

$$\sigma = (p_1, \ldots, p_k),$$

the decision variables are

$$t_0, \quad q_0, \quad s(t), \quad t_1, \ldots, t_k, \quad \delta_1, \ldots, \delta_k, \quad \psi_1, \ldots, \psi_k.$$

The route itself is also part of the search. In compact form,

$$
\begin{aligned}
\min_{\substack{\sigma,\ t_0,\ q_0,\ s(\cdot),\\ t_1,\dots,t_k,\ \delta_1,\dots,\delta_k,\ \psi_1,\dots,\psi_k}} \quad & T \\
\text{s.t.} \quad & \dot{r} = v, \\
& \dot{v} = -\mu_\odot \frac{r}{\|r\|^3} + a_{\max} s(t), \qquad s(t) \in \mathbb{S}^2, \\
& r(t_0) = r_\oplus(t_0), \\
& v(t_0) = v_\oplus(t_0) + V_0 q_0, \qquad q_0 \in \mathbb{S}^2, \\
& r(t_j) = r_{p_j}(t_j), \\
& \left\|w^+\right\| = \left\|w^-\right\|, \qquad \angle(w^+, w^-) \le \delta_{\max,j}, \\
& v^+ = v_{p_j}(t_j) + w^+, \\
& \|r(T)\| \ge R_{\text{exit}}, \qquad r(T)\cdot v(T) > 0.
\end{aligned}
$$

This is the spatial hybrid optimal-control problem: continuous thrust arcs in $\mathbb{R}^6$, discrete cone-valued flyby jumps, and a minimum-time exit target. Multiple-gravity-assist search is commonly treated as a nonlinear global optimization problem [1, 2, 3]; the continuous steering variables lead naturally to direct transcription and collocation methods [4, 5]. Figure 1 shows the corresponding route transition in a reduced spatial visualization.

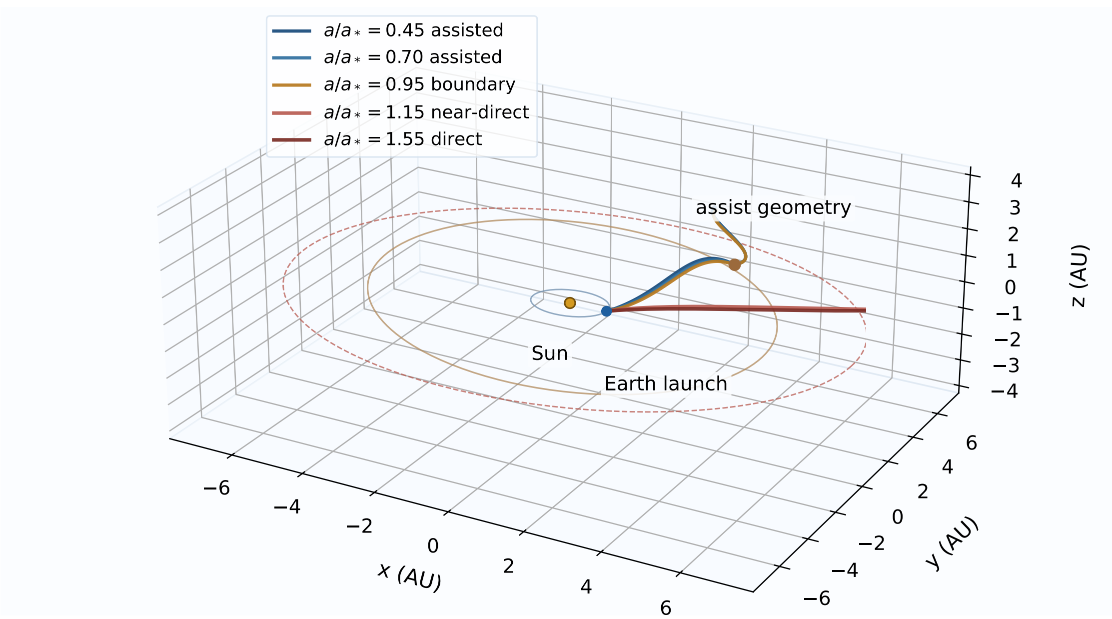

Figure 1: A reduced-model trajectory family across the switching threshold: assisted routes dominate below a_*, while near-direct escape dominates above it.

The input data are

$$
\mu_\odot, \qquad \{R_i, \mu_i, \phi_i, I_i, \Omega_i, r_{p,i}^{\min}\}_{i=1}^m,
$$
$$
V_0, \qquad a_{\max}, \qquad [t_{\min}, t_{\max}], \qquad R_{\text{exit}}, \qquad k_{\max}.
$$

The output is a best-found launch phase, launch direction, flyby route, encounter times, turn angles, flyby azimuths, thrust steering law, and exit time.

6. Energy Accounting and the Spatial Assist Threshold

Use heliocentric specific energy

$$\varepsilon = \frac{\|v\|^2}{2} - \frac{\mu_\odot}{\|r\|}.$$

Since

$$\dot{\varepsilon} = v \cdot u,$$

thrust is most effective where the spacecraft is fast and the thrust direction has a large projection onto the velocity. Write a for a candidate thrust acceleration level and let $\Lambda_\rho(\Theta_0)$ denote the average energy gain rate per unit thrust acceleration along route family ρ:

$$\dot{\varepsilon}_{\text{thrust}} \approx a\,\Lambda_\rho(\Theta_0).$$

At a flyby of planet i,

$$v^\pm = v_i + w^\pm, \qquad \|w^+\| = \|w^-\| = v_{\infty,i}.$$

Therefore

$$\Delta\varepsilon_i = \frac{\|v^+\|^2 - \|v^-\|^2}{2} = v_i \cdot (w^+ - w^-).$$

This is the usual gravity-assist energy exchange, written in heliocentric specific-energy form [6, 7]. Because

$$\|w^+ - w^-\| = 2v_{\infty,i} \sin\frac{\delta_i}{2},$$

we may write

$$\Delta\varepsilon_i = 2\eta_i\,\|v_i\|\; v_{\infty,i} \sin\frac{\delta_i}{2}.$$

For the inclined circular model,

$$\|v_i\| = \sqrt{\frac{\mu_\odot}{R_i}}.$$

The alignment factor is

$$\eta_i = \widehat{v}_i \cdot \widehat{(w^+ - w^-)}, \qquad -1 \le \eta_i \le 1.$$

In three dimensions η_i depends on the launch phase, the encounter time, the turn angle δ_i, and the flyby azimuth ψ_i. This is the main new geometric degree of freedom in the spatial model.

For a route $\sigma = (p_1, \ldots, p_k)$, define

$$G_\sigma(\Theta_0) = \sum_{j=1}^{k} 2\eta_{p_j}(\Theta_0, t_j, \delta_j, \psi_j)\sqrt{\frac{\mu_\odot}{R_{p_j}}}\, v_{\infty,j} \sin\frac{\delta_j}{2},$$

with each δ_j constrained by the flyby bound above. Let

$$E_{\text{req}}(\Theta_0; R_{\text{exit}})$$

be the energy deficit from launch to the exit sphere in the reduced energy model. One may write

$$E_{\text{req}} = \big(\varepsilon_{\text{tar}}(R_{\text{exit}}) - \varepsilon_0(\Theta_0)\big)_+, \qquad (z)_+ := \max\{z, 0\}.$$

Theorem 1: Conceptual reading.

- The theorem marks the thrust level where waiting for a useful flyby stops paying for itself. Below that level, the planet supplies enough effective energy to justify the detour; above it, onboard thrust erases the advantage.
- The proof is a bookkeeping argument between two clocks. One clock measures how much powered escape time the assist saves, and the other measures how much time is lost reaching the right spatial geometry.

Theorem 1 (Spatial phase-dependent thrust threshold)**.** *Fix a launch phase Θ_0 and a gravity-assist route σ. Suppose the direct route and the route σ are approximated by*

$$T_D(a, \Theta_0) = \frac{E_{\text{req}}(\Theta_0; R_{\text{exit}})}{a\,\Lambda_D(\Theta_0)},$$

and

$$T_\sigma(a, \Theta_0) = \tau_\sigma(\Theta_0) + \frac{\big(E_{\text{req}}(\Theta_0; R_{\text{exit}}) - G_\sigma(\Theta_0)\big)_+}{a\,\Lambda_\sigma(\Theta_0)}.$$

Here $\tau_\sigma(\Theta_0) > 0$ is the time cost of reaching the useful spatial flyby geometry, and $\Lambda_D, \Lambda_\sigma > 0$ are thrust-leverage rates. If

$$A_\sigma(\Theta_0) := \frac{E_{\text{req}}(\Theta_0; R_{\text{exit}})}{\Lambda_D(\Theta_0)} - \frac{\big(E_{\text{req}}(\Theta_0; R_{\text{exit}}) - G_\sigma(\Theta_0)\big)_+}{\Lambda_\sigma(\Theta_0)} > 0,$$

then the route-specific switching thrust is

$$a_{*,\sigma}(\Theta_0) = \frac{A_\sigma(\Theta_0)}{\tau_\sigma(\Theta_0)}.$$

For $0 < a < a_{*,\sigma}(\Theta_0)$, the assisted route is faster than the direct route. For $a > a_{*,\sigma}(\Theta_0)$, the direct route is faster than that assisted route.

Proof. Subtract:

$$T_D(a, \Theta_0) - T_\sigma(a, \Theta_0) = \frac{A_\sigma(\Theta_0)}{a} - \tau_\sigma(\Theta_0).$$

If $A_\sigma(\Theta_0) > 0$, this function has exactly one zero, namely

$$a = \frac{A_\sigma(\Theta_0)}{\tau_\sigma(\Theta_0)}.$$

The sign of the difference gives the two regimes. □

The global boundary is the lower envelope

$$\mathcal{B} = \Big\{(a, \Theta_0) : T_D(a, \Theta_0) = \min_\sigma T_\sigma(a, \Theta_0)\Big\} \subset \mathbb{R}_{\geq 0} \times \mathbb{T}^m.$$

In the spatial model this boundary is shaped not only by timing and energy, but also by the out-of-plane flyby azimuths ψ_j. Those angles decide how much of the flyby is spent changing heliocentric energy and how much is spent tilting the orbit. Figure 2 shows this as a switching surface over launch phase and flyby azimuth.

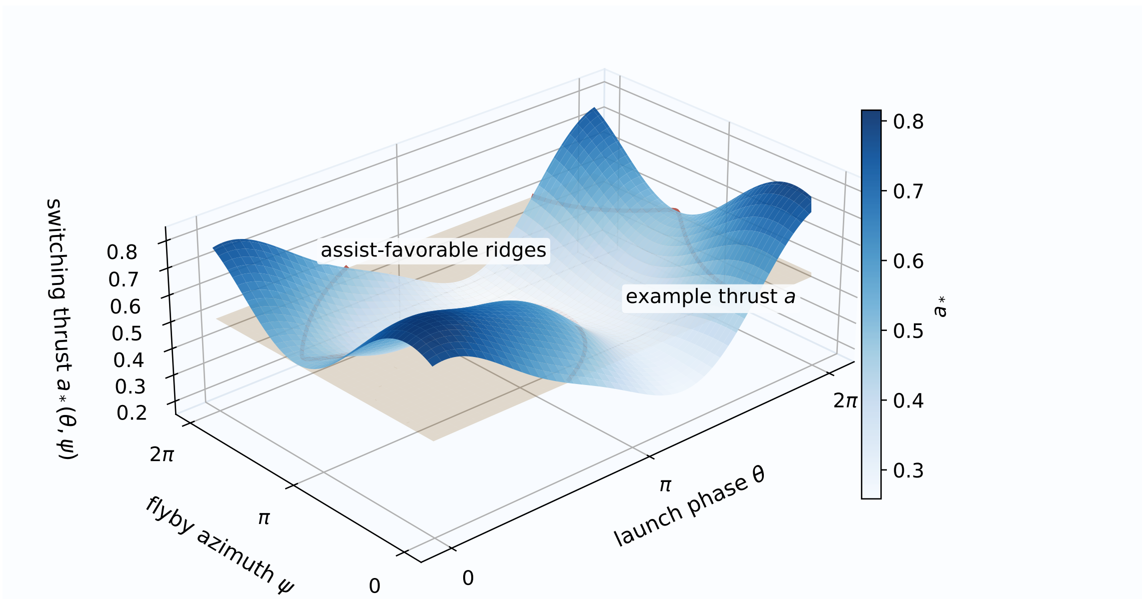

Figure 2: A conceptual switching surface $a = a_*(\Theta_0, \psi)$. Ridges are assist-favorable configurations; the translucent plane and red contour show a fixed-thrust route switch.

7. Algorithmic Consequence

A practical computation searches over routes and launch phase, then solves a continuous trajectory problem for each candidate. The state has six components, and each flyby carries both a turn angle and an azimuth variable.

1. Enumerate routes
$$\oplus \to p_1 \to \cdots \to p_k \to \text{exit}, \qquad k \le k_{\max}.$$
2. Sample launch phases $\Theta(t_0)$, encounter times, launch directions, and flyby azimuths.
3. Solve a direct-collocation problem with node variables
$$(r_n, v_n, s_n, t_n) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}.$$
4. Enforce dynamics, spatial flyby cone constraints, turn-angle bounds, and the exit condition.
5. Compare exit times and keep the best feasible trajectory.

In schematic form:

```
best = infinity
for route σ with k ≤ k_max :
  for spatial launch and timing guess:
    for flyby cone guess (δ_j, ψ_j) :
      traj = solve_collocation(σ, guess)
      if feasible and traj.T < best:
        best = traj.T
        best_traj = traj
return best_traj.
```

Thus the natural mathematical object is a hybrid optimal-control problem on

$$\mathbb{R}^6 \times \mathbb{T}^m,$$

with spatial thrust arcs, cone-valued flyby jumps, and a phase-dependent switching surface separating direct and assisted escape.

References

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