The Other Part of One

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The alternating harmonic series is an old acquaintance:

$$L := 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

It belongs to the logarithmic expansion associated with Nicolaus Mercator's *Logarithmo-technia* of 1668 [1], when infinite series were becoming tools for calculating familiar quantities. Wallis's account of Mercator's work was read to the Royal Society on 16 July 1668 [2], placing the logarithm's infinite arithmetic before a new scientific audience. A smaller question leads somewhere unexpected: what becomes of the part of one that this series leaves behind?

Write $C := 1 - L$. Pairing consecutive terms in two different ways gives

$$L = \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k)},$$
$$C = \sum_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2k+1} \right) = \sum_{k=1}^{\infty} \frac{1}{(2k)(2k+1)}.$$

Here a little nineteenth-century care is appropriate. Riemann's 1854 habilitation memoir [3] explains how rearranging a conditionally convergent series can change its sum. The memoir, written in 1854, reached print only after his death, through Dedekind's publication in 1867 [3]. We have changed no order: the first line uses the even partial sums of L, and the second uses one minus its odd partial sums. Both resulting series have positive terms and converge absolutely.

The two pieces now fit together term by term:

$$\frac{1}{(2k-1)(2k)} + \frac{1}{(2k)(2k+1)} = \frac{2}{4k^2-1}, \qquad \text{so} \qquad \boxed{\sum_{k=1}^{\infty} \frac{2}{4k^2-1} = 1.}$$

The logarithm disappears, leaving rational numbers whose sum is exactly one. The mechanism is visible already at a finite stage:

$$\sum_{k=1}^{n} \frac{2}{4k^2-1} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = 1 - \frac{1}{2n+1}.$$

This elementary telescoping identity has a probabilistic reading. Define a positive-integer-valued random variable K by

$$\mathbb{P}(K = k) := \frac{2}{4k^2-1}, \qquad k = 1, 2, \ldots.$$

The finite-sum formula supplies its entire tail:

$$\mathbb{P}(K > n) = \frac{1}{2n+1} \quad (n \geq 0), \qquad \mathbb{E}[K] = \sum_{n=0}^{\infty} \mathbb{P}(K > n) = \infty.$$

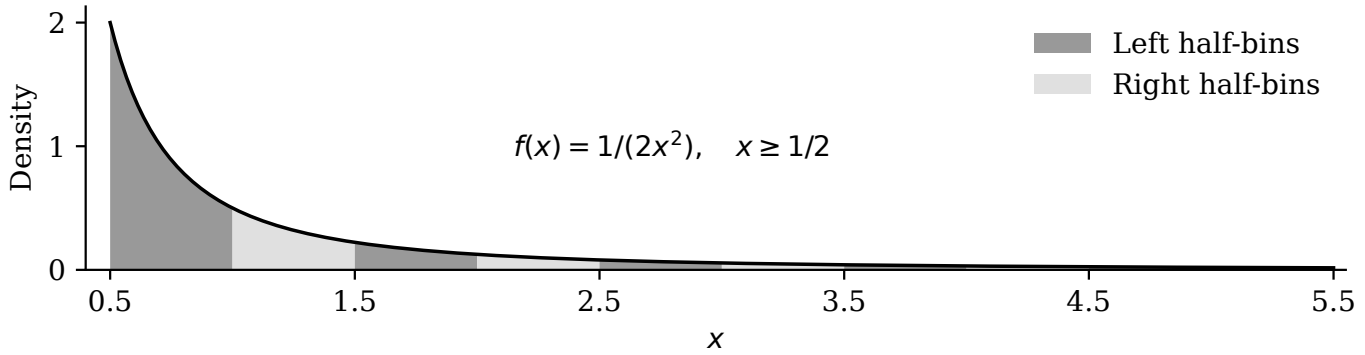
Thus a distribution obtained from a modest logarithmic identity has an infinite mean. The calculation claims no new summation theorem; its pleasure lies in the change of viewpoint. An alternating sum, its missing complement, and a law of chance occupy the same few fractions. Such examples suggest how much structure may wait within familiar things, needing only a different way of looking.

From Points to a Curve

The probabilities on the first page suggest a question of shape. Can a continuous curve carry exactly the same law? The answer is hidden in the partial fractions already before us: reciprocals are antiderivatives of inverse squares. Consider

$$f(x) = \begin{cases} \dfrac{1}{2x^2}, & x \geq \tfrac{1}{2}, \\ 0, & x < \tfrac{1}{2}. \end{cases} \qquad \int_{1/2}^{\infty} f(x)\,dx = \left[-\frac{1}{2x} \right]_{1/2}^{\infty} = 1.$$

Thus f is a probability density. It starts at height 2, falls steeply, and then stretches into an inverse-square tail. In 1897 Pareto described power-law patterns in income data [4], giving such tails an enduring place in the study of unequal shares. In modern terminology, our density is a Pareto law with scale $1/2$ and shape 1.



If X has this density, round X to the nearest integer: $K = \lfloor X + \frac{1}{2} \rfloor$. Ties have probability zero. For every $k \geq 1$,

$$\mathbb{P}(K = k) = \int_{k-1/2}^{k+1/2} \frac{dx}{2x^2} = \frac{1}{2k-1} - \frac{1}{2k+1} = \boxed{\frac{2}{4k^2-1}}.$$

Our discrete law is therefore obtained exactly by collecting the area of this curve into unit-width bins. Its tail also survives the passage between the two descriptions:

$$\mathbb{P}(X > x) = \frac{1}{2x} \quad (x \geq \tfrac{1}{2}), \qquad \mathbb{P}(K > n) = \mathbb{P}(X \geq n + \tfrac{1}{2}) = \frac{1}{2n+1} \quad (n = 0, 1, \ldots).$$

Both means are infinite; for X the reason is the divergent integral $\int_{1/2}^{\infty} x f(x)\, dx = \frac{1}{2} \int_{1/2}^{\infty} dx/x$.

Mercator's 1668 volume also advertised a "true quadrature of the hyperbola" [1]; logarithms and areas were already travelling companions. There is a final return to the logarithm. Split each bin at its integer midpoint. The left and right halves have areas

$$\int_{k-1/2}^{k} f(x)\, dx = \frac{1}{(2k-1)(2k)}, \qquad \int_{k}^{k+1/2} f(x)\, dx = \frac{1}{(2k)(2k+1)}.$$

These are precisely the two summands from the first page. If $A = \bigcup_{k \geq 1} [k - \frac{1}{2}, k)$, then

$$\boxed{\mathbb{P}(X \in A) = \ln 2, \qquad \mathbb{P}(X \notin A) = 1 - \ln 2.}$$

The darker half-bins in the figure belong to A; the lighter halves fill its complement within the support. Equal lengths need not carry equal probability: the decreasing curve gives the left halves the larger share. Mercator's logarithm reappears as the area of an alternating collection of half-intervals. What first looked like a numerical remainder now has a place under a curve. The structure was already there, quietly shared by the sum, the integral, and the law of chance.

References

[1] N. Mercator, *Logarithmo-technia, sive methodus construendi logarithmos nova, accurata, & facilis*, London, 1668. Digitized original, ETH-Bibliothek.

[2] J. Wallis, *Of Mr Mercator's Logarithmotechnia*, 1668; read 16 July. Royal Society Archives, RBO/4/1.

[3] B. Riemann, *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, habilitation memoir, 1854, §3; published by R. Dedekind, 1867. Transcription by D. R. Wilkins, Trinity College Dublin.

[4] V. Pareto, The new theories of economics, *J. Polit. Econ.* **5** (1897), 485–502. Digitized original.